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THE COHOMOLOGY RING OF A MODULE

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1. Introduction

Let G be a finite group and let K be an algebraically closed field of characteristic $p>0$. Suppose that M is a finitely generated KG-module. The purpose of this paper is to investigate the cohomology ring

$$\mathscr{E}(M)=\mathscr{E}_G(M)=\operatorname{Ext}^*_{KG}(M,M)\cong H^*(G,\operatorname{Hom}_K(M,M)).$$

Let $\mathscr{E}^t(M)=\operatorname{Ext}^t_{KG}(M,M)$. In [4] it was proved that an element of $\mathscr{E}^t(M)$ is nilpotent if and only if its restriction to every elementary abelian p-subgroup of G is nilpotent. Here we expand on this result by showing that if G is elementary abelian, then the nilpotency of an element in $\mathscr{E}^t(M)$ depends on that of the restrictions of the element to cyclic shifted subgroups, i.e. to certain cyclic subgroups of the group of units of KG. The radical of $\mathscr{E}(M)$ is then characterized in terms of restrictions to shifted subgroups. Using these results we get a new proof of a theorem of Avrunin and Scott [2] which asserts the equality of two varieties associated to M. An example of an indecomposable module M such that $\mathscr{E}(M)/\operatorname{Rad}\mathscr{E}(M)$ is not commutative is given in the last section.

For the case in which the prime p is 2, the main theorem and some of its consequences were proved in [3]. The proof for the general case, that we give here, is significantly different in that it employs a nontrivial spectral sequence argument. Although we use no results from [2], its necessary to adopt the Avrunin-Scott point of view. Specifically we regard KG as a truncated polynomial ring or equivalently as the restricted enveloping algebra of a commutative Lie algebra. The resulting Hopf algebra structure yields a cup product that commutes with restrictions to shifted subgroups. The theorem that is proved concerns only $\mathscr{E}(M)$ whose product is independent of the coalgebra structure.

Throughout the paper all KG-modules will be assumed to be finitely generated. If M is a KG-module, then $\Omega(M)$ is the kernel of the surjection $P_M\to M$ where P_M

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is a projective cover of M. For $t>1$, Ω^t is defined inductively by $\Omega^{t+1}(M)=\Omega(\Omega^t(M))$. These modules are unique up to isomorphism [7]. If H is a subgroup of G or a shifted subgroup of KG, let M_H denote the restriction of M to a KH-module.

2. Preliminaries

Throughout this section $G=\langle x_1,\dots,x_n\rangle$ is an elementary abelian group of order p^n and K is an algebraically closed field of characteristic p. Given a basis $\alpha^1,\dots,\alpha^n$ of K^n, let G' be the subgroup of the group of units of KG generated by $u_1,\dots,u_n$ where

$$u_i=1+\sum_{j=1}^{n}\alpha_{ij}X_j$$

for $\alpha^i=(\alpha_{i1},\dots,\alpha_{in})$ and $X_j=x_j-1$.

Now G' is an elementary abelian group of order p^n and the inclusion map $G'\to KG$ induces an algebra isomorphism $KG'\to KG$ by linear extension (see [3]). A shifted subgroup of KG is defined to be a subgroup of the group of units of KG of the form $H=\langle u_1,\dots,u_t\rangle$ for $u_1,\dots,u_t$ as above. In particular $\alpha^1,\dots,\alpha^t$ must be linearly independent. The left module KG is free as a KH-module and

$$K(G'/H)\cong KG/KG(\operatorname{Rad} KH).$$

Using the above statement as a definition we shall write $K(G/H)$ for $K(G'/H)$ even when H is not a subgroup of G.

If $\alpha\in K^n$, $\alpha\neq 0$, then let $u_\alpha=1+\sum\alpha_iX_i$. We define the rank variety of a KG-module M to be

$$V(M)=\{0\}\cup\{\alpha\in K^n \mid M_{\langle u_\alpha\rangle} \text{ is not a free } K\langle u_\alpha\rangle\text{-module}\}.$$

It is known that $V(M)$ is a homogeneous affine variety. Its dimension is the complexity of M (see [3]). Also M is a free module if and only if $V(M)=\{0\}$.

In order to deal with cup products and restrictions to shifted subgroups effectively we must use a Hopf algebra structure that is different from the usual one. We view KG as a truncated polynomial ring $KG=K[X_1,\dots,X_n]/\langle X_1^p,\dots,X_n^p\rangle$. If M and N are KG-modules then the action of KG on $M\otimes N=M\otimes_K N$ is given by

$$X_i(m\otimes n)=(X_im)\otimes n+m\otimes X_in$$

for $m\in M$, $n\in N$. Also if $f\in\operatorname{Hom}_K(M,N)$, then

$$(X_if)(m)=X_i\cdot f(m)-f(X_im).$$

So if $u_\alpha=1+\sum\alpha_iX_i=1+U_\alpha$, then

$$U_\alpha(m\otimes n)=U_\alpha m\otimes n+m\otimes U_\alpha n, \quad\text{and}\quad (U_\alpha f)(m)=U_\alpha f(m)-f(U_\alpha m).$$

Using this coalgebra structure we get a cup product action of $\mathscr{E}(K)=\operatorname{Ext}^*_{KG}(K,K)$ on $\mathscr{E}(M)=\operatorname{Ext}^*_{KG}(M,M)$. The cup product satisfies the associative laws and $\mathscr{E}(M)$ is a finitely generated module over $\mathscr{E}(K)$ [6]. The reader should be warned that in general the cup product does depend on the coalgebra structure (see (11.3) of [3]). However we will consistently use the one given above and no problems arise. The product in $\mathscr{E}(M)$ is independent of the Hopf algebra because it can be obtained from the Yoneda splice operation on equivalence classes of long exact sequences [10].

More especially, with this coalgebra structure, for any KG-module M, we can define for $\operatorname{Ext}^*_{KG}(K,M)$ a spectral sequence with respect to a shifted subgroup H. The construction is standard and we give only a summary of it here. Suppose that

$$\cdots X_1 \xrightarrow{\partial'} X_0 \xrightarrow{\varepsilon'} K_G \to 0; \qquad \cdots Y_1 \xrightarrow{\partial''} Y_0 \xrightarrow{\varepsilon''} K_{G/H} \to 0$$

are respectively KG- and $K(G/H)$-free resolutions of the trivial module. Minimal resolutions could be chosen here. We insist that $X_0 \cong KG$, $Y_0 \cong K(G/H)$. Note that the modules Y_r are KG-modules on which the elements of H act trivially. That is, the radical of KH annihilates Y_r for all $r \geq 0$. Now form the double complex $Z=\sum Z_{r,s}$ where $Z_{r,s}=Y_r \otimes X_s$. Each $Z_{r,s}$ is a free KG-module since X_s is free. We get a free KG-resolution

$$\cdots Z_2 \xrightarrow{\partial_2} Z_1 \xrightarrow{\partial_1} Z_0 \xrightarrow{\varepsilon} K \to 0 \qquad (Z,\varepsilon)$$

where $Z_n=\sum_{r+s=n} Z_{r,s}$. The spectral sequence is defined beginning with its E_0 term, $E_0=\{E_0^{r,s}\}$, where

$$E_0^{r,s}=\operatorname{Hom}_{KG}(Z_{r,s},M) \cong \operatorname{Hom}_{K(G/H)}(Y_r, \operatorname{Hom}_{KH}(X_s,M)).$$

One must check that the usual isomorphism works with the given coalgebra structure. So E_0 is a double complex with two differentials $\delta'': E_0^{r,s} \to E_0^{r+1,s}$ and $\delta': E_0^{r,s} \to E_0^{r,s+1}$ given by

$$\delta''(f)=(-1)^{r+s+1} f \circ (\partial'' \otimes 1) \quad \text{and} \quad \delta'(f)=(-1)^{r+s} f \circ (1 \otimes \infty').$$

Let $E_0^n=\sum_{r+s=n} E_0^{r,s}$. The total differential $\delta=\delta'+\delta'': E_0^n \to E_0^{n+1}$ is (except for possible change in sign) that induced by ∂ on Z. Consequently the total homology of this complex is $\operatorname{Ext}^*_{KG}(K,M)$.

Now let $\delta': E_0^{r,s+1}$ be the zeroth differential $d_0: E_0 \to E_0$. It follows that the homology of E_0 with respect to d_0 is $E_1=\{E_1^{r,s}\}$ where

$$E_1^{r,s} \cong \operatorname{Hom}_{K(G/H)}(Y_r, \operatorname{Ext}^s_{KH}(K,M)).$$

Let $d_1: E_1^{r,s} \to E_1^{r+1,s}$ be the differential induced by δ''. Its homology is $E_2=\{E_2^{r,s}\}$ where

$$E_2^{r,s} \cong \operatorname{Ext}^r_{K(G/H)}(K, \operatorname{Ext}^s_{KH}(K,M)).$$

The higher differentials are defined in the usual fashion.

It is possible to be very specific about the definitions of the higher differentials. An element $a_0 \in E_0^{r,s}$ represents a cocycle $\tilde{a}_0$ under d_{m-1} in $E_m^{r,s}$ $(m \geq 2)$ if and only if $\delta'(a_0)=0$ and there exist $a_i \in E_0^{r+i,s-i}$, $i=1,\dots,m-1$ such that $\delta'(a_i)=-\delta''(a_{i-1})$, $i=1,\dots,m-1$. Then $d_m(\tilde{a}_0)$ is defined to be the cohomology class of $\delta''(a_{m-1})$ in $E_m^{r+m,s-m-1}$. Numerous details remain to be checked.

The spectral sequence has a filtration $\mathscr{F}$ defined on the E_0 term by $\mathscr{F}_t(E_0)=\sum_{r\geq t} E_0^{r,s}$. Hence an element in $\mathrm{Ext}^n_{KG}(K,M)$ is in $\mathscr{F}_t(\mathrm{Ext}^n_{KG}(K,M))$ if and only if it can be represented by a cocycle (with respect to the total differential) in $\sum_{t\leq r\leq n} \mathrm{Hom}_{KG}(Z_{r,s},M)$. The spectral sequence converges in the sense that

$$\mathscr{F}_t(\mathrm{Ext}^n_{KG}(K,M))/\mathscr{F}_{t-1}(\mathrm{Ext}^n_{KG}(K,M)) \cong E_\infty^{t,n-t}.$$

It may be isomorphic to the Lyndon–Hochschild–Serre spectral sequence (see [9]). Whether it is or not, it does share the following very important properties with the L–H–S sequence.

Proposition 2.1. *The kernel of the restriction map* $\mathrm{res}_{G,H}:\mathrm{Ext}^n_{KG}(K,M)\to \mathrm{Ext}^n_{KH}(K,M)$ *is exactly* $\mathscr{F}_1(\mathrm{Ext}^n_{KG}(K,M))$. *The image of the inflation map* $\mathrm{inf}_{G/H,G}:\mathrm{Ext}^n_{K(G/H)}(K,M)\to \mathrm{Ext}^n_{KG}(K,M)$ *is* $\mathscr{F}_n(\mathrm{Ext}^n_{KG}(K,M))$. *The cup product respects the filtration in the sense that*

$$\mathscr{F}_t(\mathrm{Ext}^*_{KG}(K,M))\cdot \mathscr{F}_u(\mathrm{Ext}^*_{KG}(K,N)) \subseteq \mathscr{F}_{t+u}(\mathrm{Ext}^*_{KG}(K,M\otimes N))$$

for all KG-modules M and N.

The last statement follows from the fact that the cup product is defined by composition with a chain map $\mu:(Z,\varepsilon)\to(Z\otimes Z,\varepsilon\otimes\varepsilon)$ which lifts the identity homomorphism on K. However such a chain map can be defined by taking the tensor product of chain maps $\mu':(X,\varepsilon')\to(X\otimes X,\varepsilon'\otimes\varepsilon')$ and $\mu'':(Y,\varepsilon'')\to(Y\otimes Y,\varepsilon''\otimes\varepsilon'')$ and making an appropriate change in signs.

Before proceeding further we need to establish some facts about $\mathscr{E}(K)$. The following is well known and its proof is left to the reader.

Lemma 2.2. *The first two terms of a minimal projective KG-resolution of K have the form*

$$F_1 \xrightarrow{\partial_1} KG \xrightarrow{\varepsilon} K \to 0$$

where ε *is the augmentation map,* F_1 *is a free KG-module with KG-basis* $a_1,\dots,a_n$ *and* $\partial_1(a_i)=X_i\subseteq KG$. *The kernel of* ∂_1, *which is isomorphic to* $\Omega^2(K)$, *is generated by the elements*

$$b_i=X_i^{p-1}a_i,\quad i=1,\dots,n$$

and

$$c_{ij}=X_j a_i - X_i a_j,\quad 1\leq i<j\leq n.$$

That is the elements $b_1,\dots,b_n,c_{1,2},\dots,c_{n-1,n}$ *form a basis for* $\mathrm{Ker}\,\partial_1$ *modulo its radical.*

Since K is an irreducible KG-module, $\mathrm{Ext}^n_{KG}(K,K) \cong \mathrm{Hom}_{KG}(\Omega^n(K),K)$. For suppose that

$$\cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} K \to 0$$

is a minimal projective resolution of K. Because $\partial(F_{n+1}) \subseteq \mathrm{Rad}\, F_n$, any homomorphism from F_n to K is necessarily a cocycle, and all coboundaries are zero. Hence each cohomology class is uniquely represented by a cocycle which factors through the quotient $F_n \to F_n/\partial F_{n+1} \cong \Omega^n(K)$. With the notation of Lemma 2.1, let $\zeta_i \in \mathrm{Ext}^2_{KG}(K,K)$ be defined by $\zeta_i(b_j)=\delta_{ij}$, $\zeta_i(c_{kj})=0$. The elements $\zeta_1,\dots,\zeta_n$ together with the identity element $1 \in \mathrm{Ext}^0_{KG}(K,K)$ generate a polynomial subring $P(G,K)$ of $\mathscr{E}(K)$. Let $\eta_i \in \mathrm{Ext}^1_{KG}(K,K)$ be the element represented by the cocycle $h_i : F_1 \to K$ defined by $h_i(a_j)=\delta_{ij}$. The following is well known.

Proposition 2.3. *If $p>2$, then as rings*

$$\mathscr{E}(K) \cong P(G,K) \otimes \Lambda$$

where Λ is the exterior algebra generated by 1 *and $\eta_1,\dots,\eta_n$.*

If $p=2$, then $\mathscr{E}(K)$ is a polynomial ring generated by $\eta_1,\dots,\eta_n$. In the latter case $\zeta_i=\eta_i^2$.

Suppose that H is a maximal subgroup of G. The classical definition of the Bockstein element $\beta_H \in \mathrm{Ext}^2_{KG}(K,K)$ is that it is the image under the Bockstein map of the cohomology class in $H^1(G,K)$ of a suitably chosen non zero cocycle $v : G \to G/H \cong Z/pZ \subseteq K$. It can also be defined as the image under the inflation map

$$\mathrm{inf} : \mathrm{Ext}^*_{K(G/H)}(K,K) \to \mathrm{Ext}^*_{KG}(K,K)$$

of a 'canonical generator' in $\mathrm{Ext}^2_{K(G/H)}(K,K)$. When H is a shifted subgroup of KG, a Bockstein element β_H can be defined using the second definition. In any case β_H is contained in $E_\infty^{2,0}$ term of the spectral sequence, and in fact is the image under the map $E_2^{2,0} \to E_\infty^{2,0}$ of the chosen generator $b \in E_2^{2,0} \cong \mathrm{Ext}^2_{K(G/H)}(K,K)$. The choice of the generator is not really important since a different choice will change β_H only by a nonzero scalar multiple.

Proposition 2.4. *Let H be a maximal shifted subgroup of KG. Then $\beta_H \in P(G,K)$.*

Proof. Let $H=\langle u_1,\dots,u_{n-1}\rangle$ where $u_i=1+U_i$, $U_i=\sum \alpha_{ij}X_j$. Choose $(\alpha_{n1},\dots,\alpha_{nn})$ so that $\mathrm{Det}\, A=1$ where $A=(a_{ij})$. Let

$$A^{-1}=(\gamma_{ij}), \qquad u_n=1+U_n, \qquad U_n=\sum \alpha_{nj}X_j.$$

For a KG-projective resolution of K, we use the one given in Lemma 2.2. The $K(G/H)$-resolution is given by

$$E_1 \xrightarrow{\partial'} E_0 \xrightarrow{\varepsilon'} K \to 0$$

where $E_0 \cong E_1 \cong K(G/H)$, ε' is the augmentation map and ∂' is multiplication by U_n. Let $e_0 \in E_0$, $e_1 \in E_1$ be generators chosen so that $\varepsilon(e_0)=1$, $\partial'(e_1)=U_n e_0$. Then the kernel of ∂' is generated by $U_n^{p-1}e_1$ and is isomorphic to K. Our chosen generator of $\mathrm{Ext}^2_{K(G/H)}(K,K)$ is the cohomology class of the cocycle $f:\Omega^2(K_{(G/H)})\to K$ defined by $f(U_n^{p-1}e_1)=1$.

The chain map $\mu:(F,\varepsilon)\to(E,\varepsilon')$ can be described as follows. Let $\mu_0(1)=e_0$. The kernel of μ_0 is generated by all U_i, $1\le i\le n$. For each i, let $a_i'=\sum \alpha_{ij}a_j$. Then $\partial(a_i')=U_i$. So let $\mu_1:F_1\to E_1$ be defined by $\mu_1(a_i')=0$ if $i\ne n$, and $\mu_1(a_n)=e_1$. Note that E_0, E_1 are KG-modules on which H acts trivially and μ_0,μ_1 are KG-homomorphisms. It is easy to check that μ commutes with the boundary maps. Let

$$\tilde{\mu}:\Omega^2(K)\to\Omega^2(K_{(G/H)})$$

be the induced map on kernels. The inflation of the chosen generator θ is the class of the composition $f\circ\tilde{\mu}$. To prove the proposition it is only necessary to show that $\tilde{\mu}(c_{ij})=0$ for all $1\le i<j\le n$.

Now

$$a_i=\sum_k\left(\sum_j \gamma_{ij}\alpha_{jk}\right)a_k=\sum_j \gamma_{ij}a_j'.$$

So $\mu_1(a_i)=\gamma_{in}e_1$. Also

$$X_i=\sum_k\left(\sum_j \gamma_{ij}\alpha_{jk}\right)X_k=\sum_j \gamma_{ij}U_j.$$

In particular since $U_je_1=0$ whenever $j<n$, $X_i\cdot e_1=\gamma_{in}U_ne_1$. Therefore

$$\mu_1(c_{ij})=X_j\gamma_{in}e_1-X_i\gamma_{jn}e_1=0.$$

This proves the proposition. $\square$

Although we will not require this fact it should be noted that the coefficients $\sigma_1,\dots,\sigma_n$ in the expression $\beta_H=\sum \sigma_i\zeta_i$ are effectively computable. The spectral sequence tells us that $\mathrm{res}_{G,H}(\beta_H)=0$ and hence $\mathrm{res}_{G,\langle u_i\rangle}(\beta_H)=0$ for all i. But by Proposition 2.20 of [3], $\mathrm{res}_{G,\langle u_i\rangle}(\beta_H)$ is exactly $\sigma_1\alpha_{i1}^p+\cdots+\sigma_n\alpha_{in}^p$ times the canonical generator of $\mathrm{Ext}^2_{K\langle u_i\rangle}(K,K)$. If we also insist that $\mathrm{res}_{G,\langle u_n\rangle}(\beta_H)$ be the chosen generator, then $\sigma_1,\dots,\sigma_n$ is the unique solution to the set of equations

$$\sum_{j=1}^n \alpha_{ij}^p\sigma_j=\delta_{in},\quad i=1,\dots,n.$$

Hence $\sigma_j=\gamma_{jn}^p$.

Of course β_H depends on the choice of U_n but only up to nonzero scalar multiple. The choice that makes $\mathrm{Det}\,A=1$ is natural in the sense that it makes

$$\tilde{G}'=\sum_{g\in G'}g=\prod_i U_i^{p-1}=\prod_i X_i^{p-1}=\tilde{G}$$

where $G'=\langle u_1,\dots,u_n\rangle$.

The following is Alperin and Even's generalization of a lemma due to Quillen and Venkov [11]. The proof in [1] is correct even in the case when H is a maximal shifted subgroup and the alternate coalgebra structure is used.

Proposition 2.5 [1]. *Let H be a maximal shifted subgroup of KG and let β_H be the corresponding Bockstein element. For a KG-module M, let $\mathcal{F}$ be the filtration on $\mathrm{Ext}^*_{KG}(M,M)$ arising from the spectral sequence with respect to H. Then multiplication by β_H induces a homomorphism from $\mathcal{E}(M)$ onto $\mathcal{F}_2(\mathcal{E}(M))$.*

We will also require the following result.

Lemma 2.6. *Let G be any finite group. Suppose that M is a KG-module and that $\zeta \in \mathrm{Ext}^n_{KG}(K,K)$ where n is even if $p>2$. Let $I \in \mathrm{Ext}^0_{KG}(M,M)$ be the identity element. Then the cup product $\zeta I \in \mathrm{Ext}^n_{KG}(M,M)$ is in the center of $\mathcal{E}(M)$.*

Proof. Throughout the proof we use the standard isomorphism $\mathcal{E}(M) \cong \mathrm{Ext}^*_{KG}(K, \mathrm{Hom}_K(M,M))$. Let

$$\cdots \to F_1 \xrightarrow{\partial} F_0 \xrightarrow{\varepsilon} K \to 0$$

be a projective KG-resolution of K. In the usual way we can form the tensor product of the resolution with itself to get the resolution

$$\cdots \to X_1 \xrightarrow{\partial'} X_0 \xrightarrow{\varepsilon\otimes\varepsilon} K \to 0$$

where

$$X_n = \sum_{r+s=n} F_r \otimes F_s \quad \text{and} \quad \partial'(x_r \otimes x_s) = \partial x_r \otimes x_s + (-1)^r x_r \otimes \partial x_s,$$

for $x_i \in F_i$, $i=r,s$. Also there exists a chain homomorphism $\mu:(F,\varepsilon)\to(X,\varepsilon\otimes\varepsilon)$ such that the diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & K & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \mu_0} & & \| & & \\
\cdots & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & K & \longrightarrow & 0
\end{array}$$

commutes.

Let $\gamma: F_n \to K$ be a cocycle that represents ζ. Then ζI is represented by the cocycle $\gamma': F_n \mathrm{Hom}_K(M,M)$ where $\gamma'(a)=\gamma(a)I$ for $a \in F_n$. Let θ be an element of $\mathrm{Ext}^r_{KG}(K, \mathrm{Hom}_K(M,M))$ and suppose that θ is represented by

$$\theta': F_m \to \mathrm{Hom}_K(M,M).$$

Then $(\zeta I)\theta$ is represented by $\psi : F_{n+m} \to \mathrm{Hom}_K(M,M)$ where $\psi = \sigma \circ (\gamma' \otimes \theta') \circ \mu_{m+n}$ and

$$\sigma : \mathrm{Hom}_K(M,M) \otimes \mathrm{Hom}_K(M,M) \to \mathrm{Hom}_K(M,M)$$

is the composition pairing.

Let $\nu : (X, \varepsilon \otimes \varepsilon) \to (X, \varepsilon \otimes \varepsilon)$ be the chain homomorphism given by

$$\nu(x_r \otimes x_s) = (-1)^{rs} x_s \otimes x_r \quad \text{for } x_r \in F_r, x_s \in F_s.$$

It can be easily seen that $\partial' \nu = \nu \partial'$ and that ν lifts the identity on K. Consequently $\nu\mu : (F, \varepsilon) \to (X, \varepsilon \otimes \varepsilon)$ is also a chain map and $(\zeta I)\theta$ is also represented by the cocycle $\psi' = \sigma \circ (\gamma' \otimes \theta') \circ \nu \circ \mu$. However I commutes with every element of $\mathrm{Hom}_K(M,M)$ and, because of the hypothesis on the degree of ζ, $\sigma \circ (\gamma' \otimes \theta') \circ \nu = \sigma \circ (\theta' \otimes \gamma')$. Since the cocycle $\sigma \circ (\theta' \otimes \gamma') \circ \mu$ represents $\theta(\zeta I)$ we are finished.

3. Nilpotent elements in $\mathscr{E}(M)$

Throughout this section $G = \langle x_1, \dots, x_n \rangle$ is an elementary abelian group of order p^n. We keep the same notation as in the previous section. The following is the main theorem of this paper.

Theorem 3.1. *Let M be a KG-module and suppose that $\theta \in \mathrm{Ext}^t_{KG}(M,M)$, $t \geq 0$. Then θ is nilpotent if and only if $\mathrm{res}_{G,\langle u_\alpha \rangle}(\theta)$ is nilpotent for every $\alpha \neq 0$ in $V(M)$. Moreover if θ is nilpotent, then there exists an integer $q = q(n, t, \mathrm{Dim}\, M)$, depending only on n, t, and $\mathrm{Dim}\, M$, such that $\theta^q = 0$.*

The 'only if' part of the first statement is obvious because the restriction map is a homomorphism. The theorem is clearly true when $t = 0$. For the case in which $n = 1$, i.e. G is cyclic, we refer the reader to the next section. The structure of $\mathscr{E}(M)$ is written out explicitly in Proposition 4.1. So in this case, the first statement of the theorem holds simply because the restriction map is an isomorphism, while the second is true because B_1 (see (4.1)) is a finite-dimensional algebra.

The proof of the theorem requires the following fact which we state is a general context.

Proposition 3.2. *Let K be an algebraically closed field. let $R = K[\zeta_1, \dots, \zeta_n]$ be a polynomial ring in n variables, $n \geq 2$, graded by degrees. That is, let R_1 be the K-linear span of $\zeta_1, \dots, \zeta_n$. Then $R = K \oplus R_1 \oplus R_1^2 \oplus \cdots$. Let $A = \sum_{n \geq 0} A_n$ be a finitely generated graded R-module. We say that an element $l \in A_t$ satisfies property $P_R(A)$ if for any nonzero element $r \in R_1$ there exists an element $l' \in A_{t-1}$ such that $rl' = l$. If $l \in A_t$ satisfies $P_R(A)$, then there exist nonzero $r_1, \dots, r_k \in R_1$ such that $k < \mathrm{Dim}\, A_t$ and $r_1 \cdots r_k l = 0$.*

Proof. Let S be the subring $S=R[\zeta_1,\zeta_2]$. Note that l satisfies $P_S(A)$ and also satisfies condition $P_S(A')$ where $A'=\sum_{n\geq t-1}A_n$. Let $B=\sum_{n>t}A_n$ and let

$$\varphi : A' \to A'/B \cong C$$

be the quotient map. Clearly φ is an injection on A_{t-1} and A_t. In fact, as K-vector spaces $C\cong A_{t-1}\oplus A_t$. Moreover $S_1^2\cdot C=\{0\}$ and $\varphi(l)$ satisfies $P_S(C)$.

Assume that C is indecomposable as an S-module. A complete classification of finite-dimensional indecomposable S-modules M with $S_1^2 M=\{0\}$ is given in [8, Proposition 5], and C must be isomorphic to one of these. It is an easy exercise to show that since C contains the element $\varphi(l)\neq 0$ satisfying $P_S(C)$, then C cannot be of type (iv) in the classification scheme of [8]. Therefore it can be shown that one of the two following cases must occur.

Case I. There exist nonzero elements $r_1,r_2\in S$ and bases $\{a_1,\dots,a_m\}$, $\{a_1',\dots,a_m'\}$ of A_{t-1} and A_t, respectively, such that

$$r_1a_i=a_i'; \qquad r_2a_i=a_{i+1}', \quad 1\leq i\leq n-1; \qquad r_2a_m=0.$$

This situation corresponds to cases (i) and (ii) of the classification scheme. Note that there is a misprint in case (ii) of [8]. Case (i) of [8] reduces to our Case I because K is algebraically closed and every irreducible polynomial in $K[x]$ has degree one.

Case II. There exist nonzero elements $r_1,r_2\in S_1$ and bases $\{a_1,\dots,a_{m+1}\}$, $\{a_1',\dots,a_m'\}$ of A_{t-1} and A_t, respectively, such that

$$r_1a_1=0; \qquad r_1a_i=a_{i-1}', \quad 1<i\leq m;$$

$$r_2a_i=a_i', \quad 1\leq i<m; \qquad r_2a_{m+1}=0.$$

This corresponds to case (iii) of the scheme.

We claim that in either case $r_2^m a_j'=0$ for all $j=1,\dots,m$. For in situation I

$$r_2a_j'=r_2r_1a_j=r_1r_2a_j=r_1a_{j+1}', \quad \text{and} \quad r_2^m a_j'=r_1^{m-j+1}r_2^j a_m=0.$$

In Case II,

$$r_2a_j'=r_2r_1a_{j+1}=r_1a_{j+1}' \quad \text{and} \quad r_2^m a_j'=r_1^{m1j+1}r_2^j a_{m+1}=0.$$

Therefore r_2^m annihilates A_t and the theorem is correct if C is indecomposable as an S-module.

Suppose that $C=C_1\oplus\cdots\oplus C_k$ where each C_i is an indecomposable S-module. Let $C_j'=\varphi^{-1}(C_j)$. For every j, $B\subseteq C_j'$. Moreover

$$A_t=(A_t\cap C_1')\oplus\cdots\oplus(A_t\cap C_k').$$

Let $m_j=\mathrm{Dim}(A_t\cap C_j')$, and let $l=l_1+\cdots+l_k$ where $l_j\in C_j'\cap A_t$. Since $S_1(A_{t-1}\cap C_j')\subseteq A_t\cap C_j'$, we must have that l_j satisfies condition $P_S(C_j')$. Moreover $C_j'/B=C_j$ is an indecomposable S-module. Therefore for each $j=1,\dots,k$, there exists $r_j\in S_1$ such that $r_j^{m_j}l_j=0$. Note that if $l_j=0$, then r_j can be any element of S_1.

Hence

$$r_1^{m_1} \cdots r_k^{m_k} l = 0,$$

and since $m_1 + \cdots + m_k = \operatorname{Dim} A_t$ the proof of the proposition is complete. □

Proof of Theorem 3.1. Suppose that $\theta \in \operatorname{Ext}^k_{KG}(M, M)$ has the property that $\operatorname{res}_{G,\langle u_\alpha\rangle}(\theta)$ is nilpotent for every $\alpha \in V(M)$. By the remarks following the statement of the theorem, we need only show that θ is nilpotent and we can assume that $t > 0$ and $n > 1$. By induction on $|G|$ there exists $q = q(n-1, t, \operatorname{Dim} M)$ such that $\operatorname{res}_{G,H}(\theta^q) = 0$ for all maximal shifted subgroups H of KG. For notational convenience, replace θ^q by θ so that $\operatorname{res}_{G,H}(\theta) = 0$.

Let ζ be a nonzero homogeneous element of degree 2 in $P(G, K) = K[\zeta_1, \ldots, \zeta_n]$. Then $\zeta = \sum \beta_i \zeta_i$ for some $\beta_i \in K$. Let

$$W = \{\alpha \in K^n \mid \operatorname{res}_{G,\langle u_\alpha\rangle}(\zeta) = 0\}.$$

Because ζ is a linear polynomial in $\zeta_1, \ldots, \zeta_n$, W is a linear subspace of K^n of dimension $n-1$ (see the remark following the proof of Proposition 2.4). Let $\alpha^1, \ldots, \alpha^{n-1} \in K^n$ be a basis for W, and let $H = \langle u_1, \ldots, u_{n-1}\rangle$ where

$$u_i = 1 + \sum_{j=1} \alpha_{ij}(x_j - 1) \quad \text{for } \alpha^i = (\alpha_{i1}, \ldots, \alpha_{in}).$$

Then H is a maximal shifted subgroup in KG and ζ is a scalar multiple of β_H. Let $\mathscr{F}$ be the filtration arising from the spectral sequence with respect to H. Then by Proposition 2.1, $\theta \in \mathscr{F}_1(\mathscr{E}(M))$ and $\theta^2 \in \mathscr{F}_2(\mathscr{E}(M))$. By Proposition 2.5, there exists $\theta' \in \operatorname{Ext}^{2t-2}_{KG}(M, M)$ such that $\zeta\theta' = \theta^2$. Therefore Proposition 3.2, with $R = P(G, K)$ and $A = \sum_{n \geq 0} \operatorname{Ext}^{2n}_{KG}(M, M)$ is applicable, and there exist nonzero homogeneous $\gamma_1, \ldots, \gamma_s \in P(G, K)$ of degree 2 with $\gamma_1 \cdots \gamma_s \theta^2 = 0$ and $s \leq \operatorname{Dim} \operatorname{Ext}^{2t}_{KG}(M, M)$.

Let I denote the identity homomorphism in $\operatorname{Ext}^0_{KG}(M, M)$. We know from Lemma 2.6 that if $\gamma \in P(G, K)$, then γI is in the center of $\mathscr{E}(M)$. For each $j = 1, \ldots, s$, there exists $\theta_j \in \operatorname{Ext}^{2t-2}_{KG}(M, M)$ such that $\gamma_j \theta_j = \theta^2$. Hence

$$\begin{aligned}
\theta^{2s+2} &= \theta^2(\gamma_1\theta_1) \cdots (\gamma_s\theta_s) \\
&= \theta^2(\gamma_1 I)\theta_1 \cdots (\gamma_s I)\theta_s \\
&= (\gamma_1 I) \cdots (\gamma_s I)\theta^2\theta_1 \cdots \theta_s \\
&= \gamma_1\gamma_2 \cdots \gamma_s \theta^2 \theta_1 \cdots \theta_s = 0.
\end{aligned}$$

This proves the first statement of the theorem. The second follows from the fact that

$$\operatorname{Dim} \operatorname{Ext}^{2t}_{KG}(M, M) \leq (\operatorname{Dim} \Omega^{2t}(M))(\operatorname{Dim} M)$$

and

$$\operatorname{Dim} \Omega(M) \leq |G| \operatorname{Dim} M.$$

By successively applying the second inequality we get that $2s+2$ is bounded by a function of $|G| = p^n, t$, and $\operatorname{Dim} M$. □

Suppose that G is any finite group and M is a KG-module. let $\mathscr{E}\mathscr{A}(G)$ denote the set of elementary abelian p-subgroups of G. In [4] it was shown that an element $\zeta \in \mathrm{Ext}^m_{KG}(M,M)$ is nilpotent if and only if $\mathrm{res}_{G,E}(\zeta)$ is nilpotent for all $E \in \mathscr{E}\mathscr{A}(G)$. This fact can be used to characterize the radical of $\mathscr{E}(M)$.

For each $E \in \mathscr{E}\mathscr{A}(G)$ choose a set of generators $x_1,\dots,x_n$ where $|E| = p^n$, such that for each $\alpha \in K^n$ we get a cyclic shifted subgroup $\langle u_\alpha\rangle$, $u_\alpha = 1 + \sum \alpha_i(x_i - 1)$. Then we may define a rank variety $V_E(M) = V_E(M_E)$. For each $E, \alpha \in V_E(M)$ let

$$R_{E,\alpha} = \mathrm{res}_{E,\langle u_\alpha\rangle}(\mathrm{res}_{G,E}(\mathscr{E}(M)))$$

and let $S_{E,\alpha}$ be the kernel of the composition

$$\mathscr{E}(M) \xrightarrow{\mathrm{res}_{G,\langle \mu_\alpha\rangle}} R_{E,\alpha} \to R_{E,\alpha}/\mathrm{Rad}\, R_{E,\alpha}.$$

Theorem 3.3.

$$\mathrm{Rad}(\mathscr{E}(M)) = \bigcap_{E \in \mathscr{E}\mathscr{A}(G)} \left(\bigcap_{\alpha \in V_E(M)} S_{E,\alpha} \right)$$

and $\mathrm{Rad}(\mathscr{E}(M))$ *is a graded nilpotent ideal in* $\mathscr{E}(M)$.

Proof. The theorem is a generalization of Theorem 10.5 of [3] and the proofs are essentially the same. let $J = \bigcap_{E,\alpha} S_{E,\alpha}$. By Proposition 4.1 (next section), Theorem 3.1, and Theorem 3.1 of [4], J is generated by homogeneous elements and every homogeneous element in J is nilpotent. Clearly J must contain the radical of $\mathscr{E}(M)$ since each restriction map $\mathscr{E}(M) \to R_{E,\alpha}$ is a surjection. Consequently it is sufficient to show that J is nilpotent. By Even's Theorem [6], $\mathscr{E}(M)$ is a finitely generated module over $\mathrm{Ext}^{2*}_{KG}(K,K)$ which is a Noetherian ring. So $\mathscr{E}(M)$ satisfies the ascending chain condition on left ideals. The proof of the theorem is completed by applying the following variation on the Theorem of Levitzki. □

Proposition 3.4. *Let A be a graded ring which satisfies the ascending chain condition on left ideals. Let U be a graded left ideal with the property that every homogeneous element in U is nilpotent. Then U is a nilpotent ideal.*

Proof. We proceed almost exactly as in the proof of Levitzki's Theorem in [9, p. 199]. By hypothesis there exists a finite set $\{a_1,\dots,a_n\}$ of elements which generate U. For each i there exist homogeneous elements $b_{ij} \in U$ such that $a_i = \sum_{j=1}^{n_i} b_{ij}$. Hence the set $\{b_{ij}\}$ also is a generating set for U. That is, $U = \sum A b_{ij}$ and $U^2 = \sum U b_{ij}$. Let C be the multiplicative subsemigroup of A generated by the b_{ij}'s. Then $U^k = UC^{k-1}$ for all $k > 1$. However if D is any subsemigroup of A, the left annihilator $(0:D)$ of D is a left ideal in A, and hence the multiplicative semigroup of A satisfies the ascending chain condition on objects of the form $(0:D)$. It follows from Proposition 1 on page 199 of [9] that any nil subsemigroup of A is nilpotent. In particular C is nilpotent and U is a nilpotent ideal.

4. Equality of varieties

We begin this section with a statement about the ring $\mathscr{E}_H(M)=\mathrm{Ext}^*_{KH}(M,M)$ where M is a module over a cyclic group H of order p. The result is similar to that given in section 10 of [3] for the case in which $p=2$. Although the proof is somewhat more complicated when p is odd, it is straightforward and we leave it as an exercise for the reader. When $p=2$, Proposition 4.1 is the same as Lemma 10.1 of [3] because in this case $B_1=B_2$ and all of the pairings, φ_i, given below, are the same. The coalgebra structure used here is the standard one. In particular

$$(xf)(m)=xf(x^{-1}m) \quad \text{for } x\in H, m\in M, f\in \mathrm{Hom}_K(M,M).$$

Let $H=\langle x\rangle$ be a cyclic group of order p and let M be a KH-module. Let $B_0=\mathrm{Hom}_{KH}(M,M)$. Suppose that B_0' is the set of all $f\in B_0$ such that f factors through a projective KH-module. Then $B_0'=\tilde{H}\,\mathrm{Hom}_K(M,M)$ where $\tilde{H}=\sum_{i=0}^{p-1}x^i$. Let $B_1=B_0/B_0'$ and let $\sigma: B_0\twoheadrightarrow B_1$ be the quotient map. Suppose that

$$U=\{f\in \mathrm{Hom}_K(M,M)\mid \tilde{H}f=0\}$$

and

$$V=\{(x-1)f\mid f\in \mathrm{Hom}_K(M,M)\}.$$

It is easy to see that $V\subseteq U$. Let $B_2=U/V$, and $\tau: U\to U/V$ be the natural quotient. We have bilinear pairings

$$\varphi_1: B_1\times B_1\to B_1, \qquad \varphi_2: B_1\times B_2\to B_2,$$

$$\varphi_3: B_2\times B_1\to B_2, \qquad \varphi_4: B_2\times B_2\to B_1,$$

which are defined as follows. Suppose that $f_1,f_2\in B_0$, $h_1,h_2\in U$. Then

$$\varphi_1(\sigma(f_1),\sigma(f_2))=\sigma(f_1\circ f_2),$$

$$\varphi_2(\sigma(f_1),\tau(h_1))=\tau(f_1\circ h_1),$$

$$\varphi_3(\tau(h_1),\sigma(f_1))=\tau(h_1\circ f_1),$$

$$\varphi_4(\tau(h_1),\tau(h_2))=\sigma(h_1\vee h_2)$$

where

$$(h_1\vee h_2)(m)=\sum_{i=0}^{p-1} x_i h_1\left(\sum_{j=i+1}^{p-1} x^{j-i}h_2(x^{-j}m)\right)$$

for all $m\in M$. The composition product gives a pairing $B_0\times B_0\to B_0$ and by composing with σ we also have pairings $B_0\times B_1\to B_1$, $B_0\times B_2\to B_2$, etc.

Proposition 4.1. *Let $R=\sum_{h\geq 0}R_n$ be the graded ring in which R_n consists of all (n,γ) for*

$\gamma\in B_0$ *if $n=0$,*

$\gamma\in B_1$ *if $n>0$ is even, and*

$\gamma\in B_2$ *if n is odd.*

Addition in R_n *is given by*

$$(n, \gamma_1) + (n, \gamma_2) = (n, \gamma_1 + \gamma_2).$$

The multiplication $R_n \times R_m \to R_{n+m}$ *is given by the formula*

$$(n, \gamma_1)(m, \gamma_2) = (n+m, \varphi(\gamma_1, \gamma_2))$$

where φ *is the appropriate pairing. Then* $R \cong \mathscr{E}_H(M)$ *as graded rings. Suppose that* S *is any graded subalgebra of* R *such that* S *contains an element of the form* $(2t, \gamma)$ *for* $t > 0$ *and* γ *invertible in* B_1. *Then the radical of* S *is a nilpotent graded ideal in* S.

The point of the proof is that there exists a projective KH-resolution

$$\cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} K \to 0$$

where $F_n = KHe_n \cong KH$ and $\partial_n(e_n) = (x-1)e_{n-1}$ when n is odd, $\partial_n(e_n) = \tilde{H}e_{n-1}$ if n is even. So, for example, if n is even, then the element $(n, \sigma(f)) \in R_n$ corresponds to the cohomology class in $\operatorname{Ext}^n_{KH}(K, \operatorname{Hom}_K(M, M))$ of the cocycle ψ where $\psi(e_n) = f$. The last statement is proved essentially the same way as Lemma 10.2 of [3].

Suppose now that $G = \langle x_1, \ldots, x_n \rangle$ is an elementary abelian group of order p^n. Let $H = \langle u_\alpha \rangle$ be a cyclic shifted subgroup of KG where

$$u_\alpha = 1 + \sum \alpha_i(x_i - 1), \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in K^n, \ \alpha \neq 0.$$

If M is a KG-module, then there exist KG-submodules M_0 and M_1 such that M_1 is a free KH-module, M_0 has no free submodules and $M_H = M_0 \oplus M_1$. If $n > 0$, then $\operatorname{Ext}^n_{KH}(M, M) = \operatorname{Ext}^n_{KH}(M_0, M_0)$. Suppose that $f, f' \in B_0 = \operatorname{Hom}_{KH}(M_0, M_0)$ and that $\sigma(f) = \sigma(f')$. It can be easily seen that f is invertible if and only if f' is invertible and f is nilpotent if and only if f' is nilpotent.

The proof of the following is very similar to that of Proposition 10.3 of [3], and we will not repeat it here. The only modifications necessary are those noted in the preceding paragraph. It should be emphasized again that the standard diagonal map is used here to define cup products. Since the action of $\mathscr{E}(K)$ on $\mathscr{E}(M)$ depends on the coalgebra structure, this actions does not necessarily commute with restrictions to shifted subgroups.

Proposition 4.2. *With the above notation let* ζ *be an element of degree* $2t$ *in* $P(G, K)$, *and let* $I \in \operatorname{Ext}^0_{KG}(M, M)$ *be the identity homomorphism. Suppose that* $\operatorname{res}_{G,H}(\zeta I) = (2t, \sigma(f))$ *for* $f \in B_0 = \operatorname{Hom}_K(M_0, M_0)$. *If* $\operatorname{res}_{G,H}(\zeta) \neq 0$, *then* f *must be invertible while if* $\operatorname{res}_{G,H}(\zeta) = 0$, *then* f *is nilpotent.*

For a KG-module M, let $J(M)$ denote the ideal in $P(G, K)$ consisting of all ζ such that $\zeta I = 0$. That is $J(M)$ is the annihilator in $P(G, K)$ of $\mathscr{E}(M)$. Let

$$W(M) = \{\alpha \in K^n \mid \operatorname{res}_{G, \langle u_\alpha \rangle}(\zeta) = 0 \text{ for all } \zeta \in J(M)\}.$$

Clearly $J(M)$ is a homogeneous ideal in $P(G,K)$. In [3, Proposition 2.22] it was shown that if $\zeta = f(\zeta_1, \dots, \zeta_n)$ is a homogeneous element in $\zeta_1, \dots, \zeta_n$, then

$$\mathrm{res}_{G,\langle u_\alpha\rangle}(\zeta) = f(\alpha_1^P, \dots, \alpha_n^P) \cdot \gamma$$

where γ is the canonical generator in $\mathrm{Ext}^t_{K\langle u_\alpha\rangle}(K,K)$, $t = 2\deg(f)$. So $W(M)$ is a homogeneous subvariety of K^n. It was proved in [3, Theorem 7.5] that $V(M) \subseteq W(M)$. The reverse inclusion was first proved by Avrunin and Scott using methods are different from the ones employed here.

Theorem 4.3 [2]. *Let M be a KG-module. Then $V(M) = W(M)$.*

Proof. Let $I(M) \subseteq K[\zeta_1, \dots, \zeta_n]$ be the ideal of $V(M)$. Let $\varphi : K \to K$ be the Frobenius automorphism, $\sigma(a) = a^P$. Then φ induces an automorphism of $P(G,K)$ by operating on the coefficients of the polynomials. Clearly

$$\varphi(f)(\alpha_1^P, \dots, \alpha_n^P) = [f(\alpha_1, \dots, \alpha_n)]^P.$$

Hence, by the remark preceding the theorem, the ideal of $W(M)$ is in the radical of $\sigma^{-1}(J(M))$. Let f be a homogeneous polynomial in $I(M)$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in V(M)$ we have that $f(\alpha) = 0$ and $\sigma(f)(\alpha_1^P, \dots, \alpha_n^P) = (f(\alpha))^P = 0$. Let $\zeta = (\sigma f)(\zeta_1, \dots, \zeta_n)$. By Proposition 4.2, $\mathrm{res}_{G,\langle u_\alpha\rangle}(\zeta I)$ is nilpotent for all $\alpha \in V(M)$. Hence by Theorem 3.1, ζ is in the radical ideal of $J(M)$, and $\sigma^{-1}(\zeta)$ is in the ideal of $W(M)$. This proves that $W(M) \subseteq V(M)$. □

5. Commutativity of cohomology rings

In [5] it was shown that if $G = \mathrm{SL}(2, p^n)$ and M is an irreducible KG-module, then the ring $\mathscr{E}(M)/\mathrm{Rad}\,\mathscr{E}(M)$ is commutative. It is possible that this statement is true for any finite group G and any irreducible module M. However the following theorem demonstrates that it is not true if M is only indecomposable. Let $M_n(K)$ denote the algebra of $n \times n$ matrices over K.

Theorem 5.1. *Let $G = \langle x, y\rangle$ be an elementary abelian group of order $p^2, p \geq 5$. For any positive integer n there exists an indecomposable KG-module M such that there is a K-algebra homomorphism*

$$\psi : \mathscr{E}(M) \to M_n(K)$$

which is surjective.

Proof. Let $X = x - 1$, $Y = y - 1$. Let $A = \sum_{i=1}^n KGa_i$ be the free KG-module generated by $a_1, \dots, a_n$. Let $B \subseteq A$, be the submodule generated by the elements

$$XYa_i, \quad 1 \leq i \leq n; \qquad X^{p-1}a_{j-1} - Y^2 a_j, \quad 2 \leq j \leq n-1;$$

$$Y^2 a_1; \qquad X^{p-1}a_{n-1} - Ya_n; \quad \text{and} \quad Xa_n.$$

Let $M=A/B$. Then M is generated by $c_i=a_i+B$, $i=1,\dots,n$. A K-basis for M consists the elements

$$X^j c_i, \quad i=1,\dots,n-1,\ j=0,\dots,p-1; \qquad c_n;$$

and

$$Yc_i, \quad i=1,\dots,n-1.$$

Hence $\operatorname{Dim} M=n+p(n-1)$.

Observe that if $m\in M$ and $Ym=0$, then $m\in \operatorname{Rad} KG\cdot M$. Also if $m\in(\operatorname{Rad} KG)M$, then $X^{p-1}m=0$. Suppose that $M=M_1\oplus M_2$. The socle of M is the K-linear span of the elements Yc_1, Yc_n, and Y^2c_i, $i=2,\dots,n-1$. One of the two direct summands, say M_1, contains an element of the form

$$m=\alpha_1 Yc_1+\alpha_2 Y^2c_2+\cdots+\alpha_{n-1}Y^2c_{n-1}+\alpha_n Yc_n.$$

where $\alpha_1\neq 0$. So $m=Yl$ where

$$l=\alpha_1c_1+\alpha_2Yc_2+\cdots+\alpha_nc_n.$$

Write $l=l_1+l_1'$ for $l_1\in M_1$, $l_1'\in M_2$. Now $Yl_1'=0$, so $l_1'\in(\operatorname{Rad} KG)M$. Therefore $X^{p-1}l=X^{p-1}l_1=Y^2c_2\in M_1$. Write $c_2=l_2+l_2'$ for $l_2\in M_1$, $l_2'\in M_2$. Since $Y^2l_2'=0$, $l_2'\in(\operatorname{Rad} KG)M+Kc_1+Kc_n$. So for some $\beta\in K$, $X^{p-1}l_2'=\beta x^{p-1}c_1\in M_1\cap M_2$. Therefore $\beta=0$ and $X^{p-1}l_2'=0$. This proves that $X^{p-1}c_2=Y^2c_3=X^{p-1}l_2\in M_1$. Continuing in this fashion we show that M_1 contains the entire socle of M. Therefore $M=M_1$ is indecomposable.

Let $H=\langle x\rangle$. Then $M_H=M_0\oplus M_1$ where M_0 has basis $\{c_n, Yc_i \mid i=1,\dots,n-1\}$ and M_1 has basis $\{X^j\sigma_i \mid 1\le i\le n-1, 0\le j\le p-1\}$. Clearly M_1 is a free KH-module and M_0 is a direct sum of n-copies of the trivial KH-module. In the notation of Proposition 4.1,

$$\sigma(B_0)=B_1=\operatorname{Hom}_{KH}(M_0,M_0)=\operatorname{Hom}_K(M_0,M_0)\cong M_n(K).$$

Also the product formula says that two elements of odd degree in $\mathscr{E}_H(M)$ have product 0, since H acts trivially on M_0. Therefore we have a homomorphism

$$\theta:\mathscr{E}_H(M)\to\operatorname{Hom}_K(M_0,M_0)\cong M_n(K)$$

where $\theta(n,\gamma)$ is $\sigma(\gamma)$ if n is 0, γ if $n>0$ is even, and 0 if n is odd. Let $\psi:\mathscr{E}(M)\to M_n(K)$ be the composition $\psi=\theta\circ\operatorname{res}_{G,H}$. It remains only to prove that ψ is onto.

It can be seen that there is a KG-homomorphism whose values on generators are $f(c_1)=0$, $f(c_i)=c_{i-1}$ for $2\le i\le n-1$, and $f(c_n)=Yc_{n-1}$. Then $\sigma(f):M_0\to M_0$ has the property that $\sigma(f)(Yc_1)=0$, $\sigma(f)(Yc_i)=Yc_{i-1}$ for $2\le i\le n-1$ and $\sigma(f)(c_n)=Yc_{n-1}$. We will show that there exists $h\in\operatorname{Hom}_{KH}(M,M)$ such that $(2,\sigma(h))\in\operatorname{res}_{G,H}(\mathscr{E}(M))$ and $h(Yc_1)=c_n$, $h(Yc_i)=h(c_n)=0$ for $i>1$. This is sufficient to prove the theorem since $\sigma(f),\sigma(h)$ generate $\operatorname{Hom}_K(M_0,M_0)$ as a ring.

As mentioned before $\operatorname{Ext}^2_{KG}(M,M)\cong\operatorname{Ext}^2_{KG}(K,\operatorname{Hom}_K(M,M))$ is a quotient of $\operatorname{Hom}_{KG}(\Omega^2(K),\operatorname{Hom}_K(M,M))$. In this case $\Omega^2(K)$ is generated by three elements

b_1, b_2, c such that

$$Xb_1 = Yb_2 = 0, \quad Yb_1 = X^{p-1}c, \qquad Xb_2 = Y^{p-1}c.$$

If $\gamma : \Omega^2(K) \to \mathrm{Hom}_K(M, M)$ is a KG-homomorphism, then $\mathrm{res}_{G,H}(cl(\gamma)) = (2, \sigma(\gamma(b_1)))$. Define the 2-cocycle γ by $\gamma(b_1) = h$, $\gamma(c) = g$, $\gamma(b_2) = 0$ where

$$h(Yc_1) = c_n, \qquad h(KHc_1) = 0, \qquad h(KGc_i) = 0, \quad i \geq 2,$$

and

$$g(Yc_1) = 0, \qquad g(X^i c_1) = 0, \quad 0 \leq i \leq p-2,$$

$$g(X^{p-1}c_1) = -c_n - X^{p-1}c_{n1},$$

$$g(KHc_2) = 0, \qquad g(KGc_i) = 0, \qquad 3 \leq i \leq n.$$

It can be checked that $Xh = 0$, $Yh = X^{p-1}g$, and $Y^{p-1}g = 0$. So γ is a homomorphism, and $\mathrm{res}_{G,H}(\gamma) = (2, \sigma(h))$ as desired.

Remarks. The above theorem is also true when $p = 3$. This case, however, is complicated by the fact that the element $h \in \mathrm{Hom}_K(M, M)$ is not the image under ψ of an element of degree 2, but rather of degree $2(n-1)$.

The following theorem shows that for any finite group G and any KG-module M, a simple $\mathscr{E}(M)$-module is defined by a surjection $\mathscr{E}(M) \to M_n(K)$, for some n. An interesting question is whether every such surjection must factor through the restriction map to some shifted cyclic subgroup of KG. That is, does every maximal ideal in $\mathscr{E}(M)$ contain the kernel of the restriction to some cyclic shifted subgroup.

Theorem 5.2. *Let G be any finite group and let M be an indecomposable KG-module. Then every irreducible $\mathscr{E}(M)$-module has finite dimension over K.*

Proof. Let $R' = \sum_{n \geq 0} \mathrm{Ext}_{KG}^{2n}(K, K) \subseteq \mathscr{E}(K)$ and let R be its image in $\mathscr{E}(M)$ under the map $\mathscr{E}(K) \to \mathscr{E}(M)$ given by cup product with the identity. By Lemma 2.6, R is a subalgebra of the center of $\mathscr{E}(M)$ and R is generated as a K-algebra by a finite set of elements $\{\gamma_1, \ldots, \gamma_r\}$. Moreoever, by Even's Theorem [6], $\mathscr{E}(M)$ is a finitely generated module over R. That is, $\mathscr{E}(M) = \sum_{i=1}^{s} R\theta_i$ for some $\theta_i \in \mathscr{E}(M)$.

We follow the argument on page 227 of [9]. Let f_t be the polynomial in elements of $\mathscr{E}(M)$ given by

$$f_t(x_1, \ldots, x_t) = \sum_{\sigma} \mathrm{sgn}(\sigma)\,(x_{\sigma(1)} \cdots x_{\sigma(n)})$$

where the sum is over all σ in the symmetric group S_t. Note that f_t is R-linear in any of its variables. Also if $x_i = x_j$ for $i \neq j$, then $f_t(x_1, \ldots, x_t) = 0$. Since $\mathscr{E}(M)$ is generated as an R-module by s elements, we have that f_{s+1} is identically zero on $\mathscr{E}(M)$. Therefore $\mathscr{E}(M)$ is a P.I. ring.

Let W be an irreducible $\mathscr{E}(M)$-module. Let U be the annihilator in $\mathscr{E}(M)$ of W. Then U is a two-sided ideal and $S = \mathscr{E}(M)/U$ is likewise a P.I. ring. Because S has a faithful irreducible module, namely W, it is a primitive ring. Hence by Theorem

1, page 226 of [9], the center of S is a field L and S is a finite-dimensional algebra over L. Let $d = \mathrm{Dim}_L(W)$. Then the action of S on W defines a homomorphism $\psi: S \to M_d(L)$. Now S is generated as a K-algebra by the elements $\gamma_i\theta_j + U$, $1 \le i \le r$, $1 \le j \le s$. Consequently L is generated as a K-algebra by the d^2rs entries of the matrices $\psi(\gamma_i\theta_j + U)$. That is L is a finitely generated algebra over K and it must be a finite algebraic extension of K. This proves the theorem.

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